Special course in Computer Science: Advanced Text Algorithms

Lecture 2: Pattern Matching Algorithms

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Pattern Matching Problem

- The pattern matching problem is one of the most investigated problems in text algorithms.

- Implementations of algorithms for this problem are used daily for accessing information
  - use Google search engine
  - make a data base query
  - use the “search” or “replace” commands in text editors

- Simple example of a pattern matching problem:
  - Find occurrences of the word “advanced” in the text “Special course in Computer Science: advanced text algorithms”

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Pattern Matching Problem

- A pattern is a nonempty language which does not contain the empty string:
  - a single string
  - a finite or infinite set of strings

- The pattern matching problem: Given a text and a pattern, find occurrences of the pattern within the text.

Pattern Matching Problem

- We will consider for now only the case when the pattern is a single string.

- The input for the pattern matching algorithm:
  - the pattern pat of length m
  - the text text of length n

- The output of the algorithm:
  - A boolean value: 1 if pat occurs in the text or 0 otherwise.
  - A set of positions where the pattern occurs in the given text.

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Notations and Definitions

- A period of a word w is an integer $0<p<|w|$ such that $w[i]=w[i+p]$ for all $i\in\{1, .., |w|-p\}$.
  - Example:
    - abcabc has period 3: $abc\,abc$
    - abababa has periods 2,4, and 6: $ab\,ab\,ab\,a$, $abab\,aba$, $\underbrace{ababab}\,\underbrace{a}$

- We denote by period(w) the smallest period of w.

- A word u is a border of a word v if u is both a prefix and a suffix of v.
  - Example: abaaaab

Notations and Definitions

- Just as in the case of periods, there are words which have several borders.

- Example: ababab

$ab\,abab$ $abab\,ab$

$abab\,ab$ $ab\,abab$

- Denote by Border(w) the longest nontrivial border of a word w, i.e., Border(w)≠w.

Pattern Matching Algorithms

- We will discuss two fundamental pattern matching algorithms:
  - Knuth-Morris-Pratt (KMP) algorithm
  - Boyer-Moore (BM) algorithm

- Both of them have two stages:
  - Pre-processing stage: computing some additional tables
  - Pattern-searching stage

Brute-force algorithm

- The scheme for a brute-force algorithm for the pattern-matching problem:

$$\text{for } i := 0 \text{ to } n - m \text{ do}$$
$$\text{check if } pat = text[i+1 \,..\, i+m]$$

- This algorithm is the origin for both KMP and BM algorithms.

- The actual implementation of this algorithm depends on how we implement the function “check”, e.g., scanning the text from the left, or from the right, or other method.

Brute-force algorithm

- Algorithm brute-force1

$$
\begin{aligned}
& i := 0 \\
& \text{while } i \le n - m \text{ do begin} \\
& \quad j := 0; \\
& \quad \text{while } j < m \text{ and } pat[j+1] = text[i+j+1] \text{ do} \\
& \qquad j := j+1; \\
& \quad \text{if } j = m \text{ then return}(true); \\
& \quad i := i+1; \\
& \text{end;} \\
& \text{return}(false)
\end{aligned}
$$

- the index i scans the positions of the text
- the index j scans the positions of the pattern
- if we found the pattern, then we stop
- otherwise, i.e., we have a mismatch between a character in the pattern and a character in the text, we increase i by 1 and we start comparing again pat and text.

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Brute-force algorithm

- Algorithm brute-force1

$i := 0$

while $i \leq n - m$ do begin

  $j := 0;$

  while $j < m$ and $pat[j+1] = text[i+j+1]$ do

    $j := j + 1;$

  if $j = m$ then return($true$);

  $i := i + 1;$

end;

return($false$)

Let text=aaaaabaa and pat=aaaba

- i=0

  aaab

  aaaaabaa

- i=1

   aaab

  aaaaabaa

- i=2

    aaaba

  aaaaabaa

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Brute-force algorithm

- Algorithm brute-force1

$i := 0$

while $i \leq n - m$ do begin

  $j := 0;$

  while $j < m$ and $pat[j+1] = text[i+j+1]$ do

    $j := j + 1;$

  if $j = m$ then return($true$);

  $i := i + 1;$

end;

return($false$)

- This algorithm scans both the text and the pattern from left to right.
- The function return(x) outputs x and stops the whole algorithm.
- The shifts have length 1.
- If pat=$a^n b$ and text=$a^{2n-1} b$, then the brute-force algorithm performs a quadratic number of comparisons.

The Brute-force algorithm

- The main drawback of the brute-force algorithm is that the shifts in the search have length 1.

- Moreover, after a shift of the pattern, the brute-force algorithm has forgotten all information about previously matched symbols.

- So it is possible that it re-compares a text symbol with different pattern symbols again and again.

- This leads to its worst case complexity of $\Theta(nm)$ (n: length of the text, m: length of the pattern).

The Brute-force algorithm

- This complexity is too large, and makes the use of this algorithm unpractical
  - in practice we search a given pattern within huge texts, or data bases
- Most used algorithms for pattern matching (Knuth-Morris-Pratt, Boyer-Moore) have linear time complexity and are more suitable for practical use.

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The Knuth-Morris-Pratt algorithm

- The Knuth-Morris-Pratt algorithm uses the information gained by previous symbol comparisons.

- It never compares again a text symbol that has already matched a pattern symbol.

- As a result, the complexity of the searching phase of the Knuth-Morris-Pratt algorithm is in $O(n)$.

The Knuth-Morris-Pratt algorithm

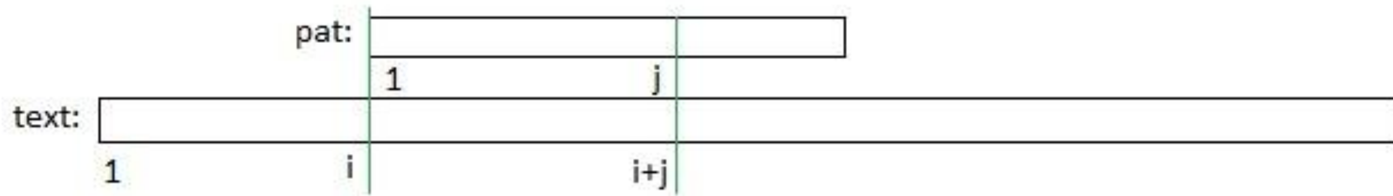
- However, a pre-processing of the pattern is necessary in order to analyze its structure.

- The pre-processing phase has a complexity of $O(m)$.

- Since $m \leq n$, the overall complexity of the Knuth-Morris-Pratt algorithm is in $O(n)$.

Morris-Pratt (MP) algorithm

- Let pat[1..j]=text[i+1..i+j] for some j≥1.

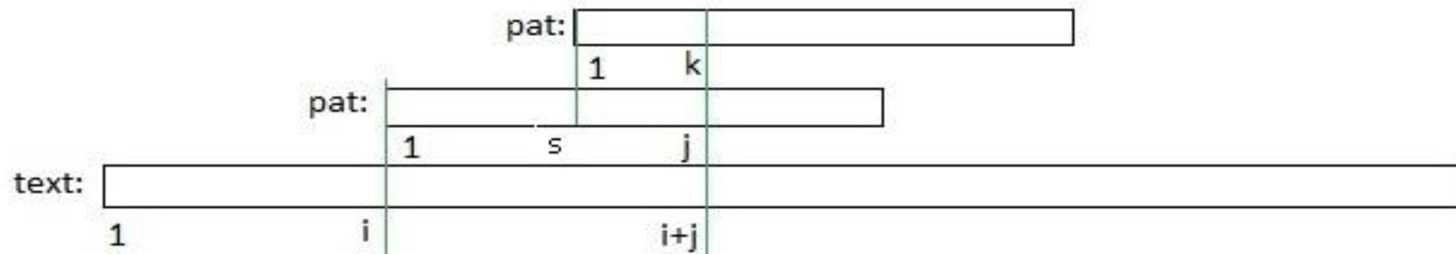


- We want to shift the search by more than 1 unit.

- Safe shift s = a positive integer s such that there is no occurrence of the pattern at positions i, i+1, .., i+s-1, but there might be one at i+s.

Safe shift

- Suppose an occurrence of the pattern starts at position i+s and let k=j-s.



- Then pat[1..k] and pat[1..j] are suffixes of the same text: text[1..i+j]

- cond(j,k): pat[1..k] is a proper suffix of pat[1..j]

- So, s is a safe shift if k=j-s is the largest integer satisfying cond(j,k)

Safe shift

- cond(j,k): pat[1..k] is a proper suffix of pat[1..j]
- Denote Bord(j)= the largest integer k satisfying cond(j,k)
- Bord(j)=|Border(pat[1..j])|
- Then, the smallest safe shift is

MP_Shift(j)=j-Bord(j)

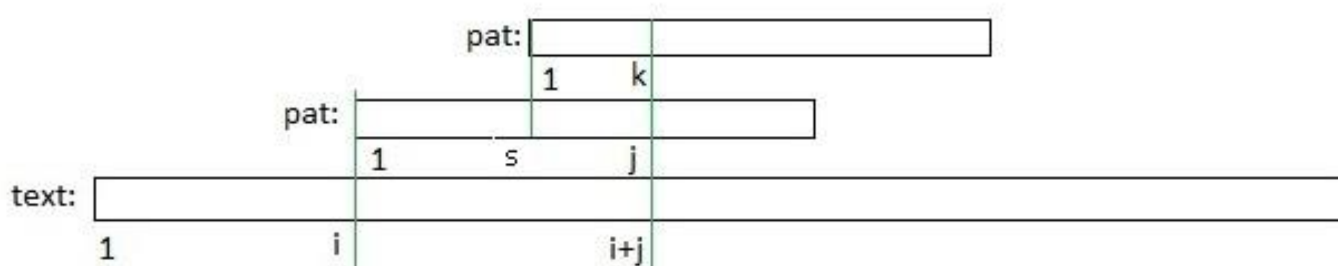


Table of borders

- The table Bord, called the table of borders, is essential for this algorithm.

- It carries the information which allows us to compute the length of a safe shift ($s=j-Bord(j)$) in the case of a mismatch.

Example of a table of borders

- Bord(j)=|Border(pat[1..j])|

- Example: Let pat=abababababb. Then,

Bord(1)=|Border(pat[1])|=|Border(a)|=0,

Bord(2)=|Border(pat[1..2])|=|Border(ab)|=0,

Bord(3)=|Border(pat[1..3])|=|Border(aba)|=1,

Bord(4)=|Border(pat[1..4])|=|Border(abab)|=2,...

Bord=[0,0,1,2,3,4,5,6,7,8,0]

- If j=0, pat[1..j]=ε, then the shift must be 1 and we define Bord(0)=-1

Computing the table of borders

- As a pre-processing step in the MP-algorithm we compute the table Bord.

procedure compute_Bord

$Bord(0) := -1; \quad t := -1;$

for $j := 1$ to m do begin

while $t \geq 0$ and $pat[t+1] \neq pat[j]$ do

$t := Bord(t);$

$t := t + 1; \quad Bord(j) := t;$

end

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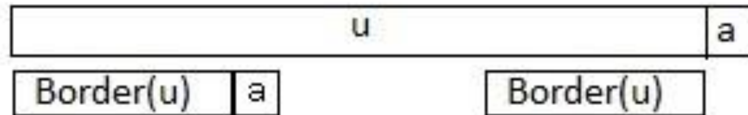
Computing the table of borders

- Lemma: For a word u and a letter a, we have

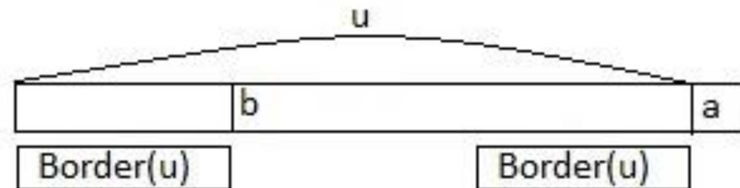
$$Border(ua) = \begin{cases} Border(u)a, & \text{if } Border(u)a \text{ is a prefix of } u, \\ Border(Border(u)a), & \text{otherwise} \end{cases}$$

Proof: (not required for the exam)

- If Border(ua) is not empty, then Border(ua)=wa, where w is a border of u
- If Border(u)a is a prefix of u, then obviously Border(ua)=Border(u)a
  - Border(u)a is a border of ua
  - It is the maximal border



- Otherwise, i.e., if Border(u)a is not a prefix of u
  - Border(ua) is a prefix of Border(u) since |Border(ua)|≤|Border(u)|+1
  - Border(ua) is also a suffix of Border(u)a
  - Border(ua) is maximal with this property.
- So, Border(ua)=Border(Border(u)a).



Computing the table of borders

procedure compute_Bord

$Bord(0) := -1; \quad t := -1;$

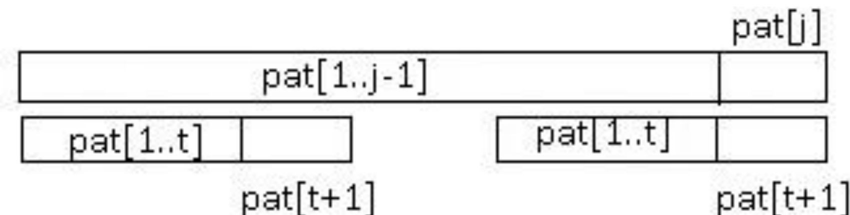
for $j := 1$ to m do begin

while $t \geq 0$ and $pat[t+1] \neq pat[j]$ do

$t := Bord(t);$

$t := t+1; \quad Bord(j) := t;$

end



- Lemma: The procedure compute_Bord applied to a string pat of length m produces the table of borders for pat.

Proof:

- The table Bord is computed sequentially by the procedure compute_Bord: from the prefix of length 1 to pat itself.
- At the beginning of each "for cycle" j, t=Bord(j-1), i.e., pat[1..t] is both a prefix and a suffix of pat[1..j-1], and is maximal with this property.
- If pat[t+1]=pat[j], i.e., the previous border can be extended by 1 character, i.e., Bord(j)=Bord(j-1)+1
- Otherwise, i.e., pat[t+1]≠pat[j], we compute recursively the new border in the while loop, according to the previous lemma.

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Computing the table of borders

procedure compute_Bord

$Bord(0) := -1;\ \ t := -1;$

for $j := 1$ to m do begin

while $t \geq 0$ and $pat[t+1] \neq pat[j]$ do

$t := Bord(t);$

$t := t+1;\ \ Bord(j) := t;$

end

Example: pat=abaababaaba, m=11

Bord(0)=-1; t=-1

j=1

t:=0; Bord(1):=0;

j=2

t≥0; pat[1]≠pat[2]

t:=Bord(0)=-1

t:=0; Bord(2)=0;

j=3

t≥0; pat[1]=pat[3]

t:=1; Bord(3):=1;

j=4

t≥0; pat[2]≠pat[4]

t:=Bord(1)=0

pat[1]=pat[4]

t:=1; Bord(4)=1;

Computing the table of borders

procedure compute_Bord

$Bord(0) := -1; \quad t := -1;$

for $j := 1$ to m do begin

while $t \geq 0$ and $pat[t+1] \neq pat[j]$ do

$t := Bord(t);$

$t := t+1; \quad Bord(j) := t;$

end

Example: pat=abaababaaba, m=11

j=5

t=1≥0; pat[2]=pat[5]

t:=2; Bord(5)=2

j=6

t≥0; pat[3]=pat[6]

t:=3; Bord(6)=3

j=7

t≥0; pat[4]≠pat[7]

t:=Bord(3)=1

t≥0; pat[2]=pat[7]

t:=2; Bord(7)=2

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Computing the table of borders

procedure compute_Bord

$Bord(0) := -1; \quad t := -1;$

for $j := 1$ to m do begin

$\quad$ while $t \geq 0$ and $pat[t+1] \neq pat[j]$ do

$\quad\quad t := Bord(t);$

$\quad t := t + 1; \quad Bord(j) := t;$

end

- Lemma: The maximum number of character comparisons executed by algorithm compute_Border is 2m-3.

Proof:

- we use a so-called "store principle" where t is seen as the number of items in store
- when j=1, we have t<0, so no character comparison is done and t becomes 0 (i.e., the store is empty at the beginning)
- for each 2≤j≤m we add at most one element in the store (we execute the statement t:=t+1). So the maximal size of the store is m-2.

Computing the table of borders

procedure compute_Bord

$Bord(0) := -1; \quad t := -1;$

for $j := 1$ to m do begin

while $t \geq 0$ and $pat[t+1] \neq pat[j]$ do

$t := Bord(t);$

$t := t+1; \quad Bord(j) := t;$

end

- each time we execute statement t:=Bord(t), t is decreased, i.e., we eliminate some elements from the store.
- So there are at most m-2 unsuccessful character comparisons pat[t+1]≠pat[j] (for each of them we execute the statemet t:=Bord(t))
- For each 2≤j≤m, there is at most one successful character comparison.
- Thus the total number of character comparisons is at most (m-2)+(m-1)=2m-3

Morris-Pratt (MP) algorithm

- Using the table of borders Bord and the safe shift we obtain an improved version of the algorithm brute-force1, i.e., the Morris-Pratt (MP) algorithm:

Algorithm MP;

$i := 0;\ \ j := 0;$

while $i \le n - m$ do begin

while $j < m$ and $pat[j+1] = text[i+j+1]$ do

$j = j + 1;$

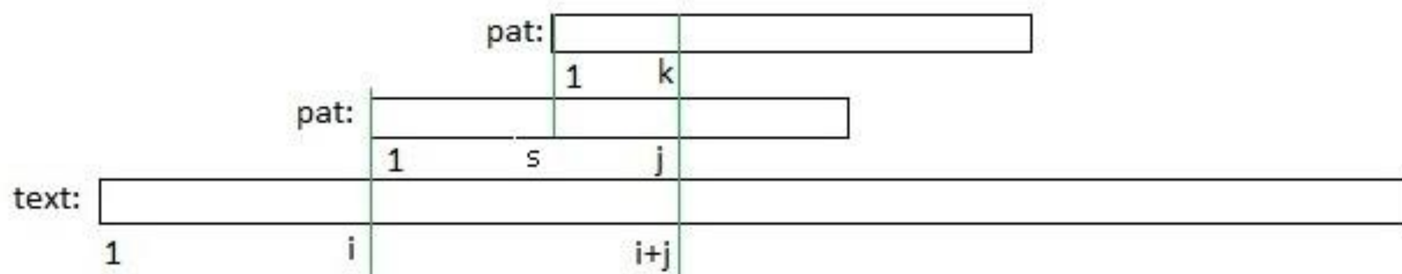
if $j = m$ then $return(true);$

$i := i + MP_Shift(j);\ \ j := \max(0, j - MP_Shift(j));$

end;

$return(false)$

Morris-Pratt (MP) algorithm



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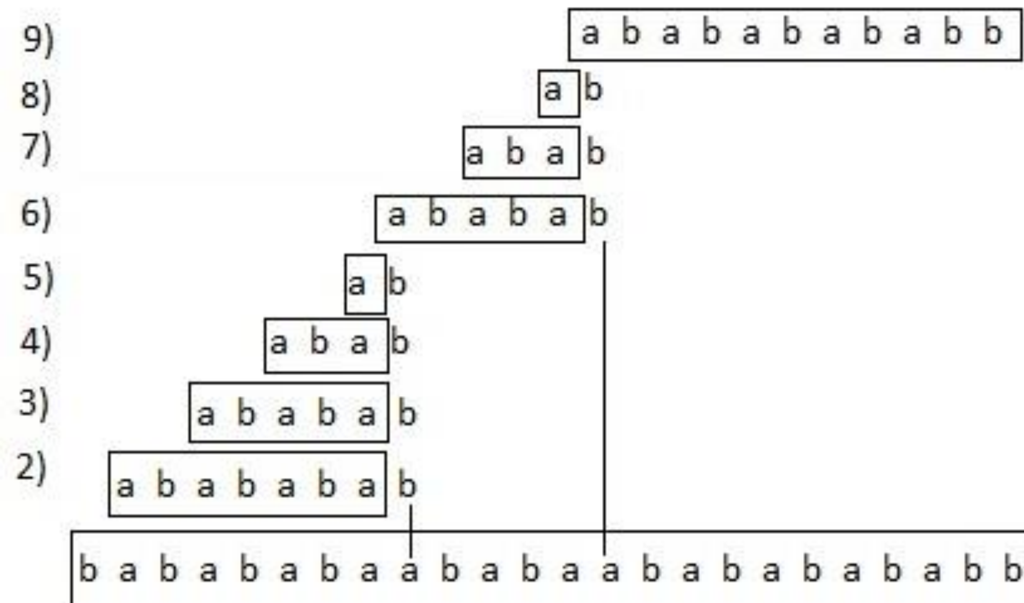
Morris-Pratt (MP) algorithm

- Example: Let text=bababababaababaababababababb and pat=ababababab b.



1) i=0, j=0 pat[1]≠text[1]

2) i=i+MP_Shift(j)=
  =i+j-Bord(j)=0+0-(-1)=1;
  j=max(0,j-MP_Shift(j))=0

pat[1]=text[2]; pat[2]=text[3]
pat[3]=text[4]; pat[4]=text[5]
pat[5]=text[6]; pat[6]=text[7]
pat[7]=text[8]; pat[8]≠text[9]
j=7, i=1

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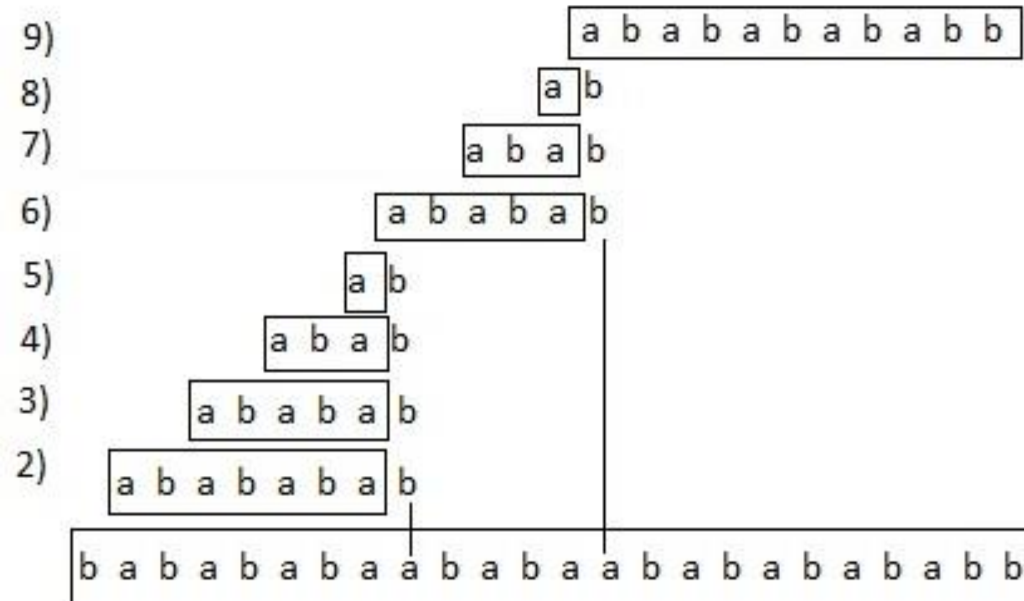
Morris-Pratt (MP) algorithm

- Example: Let text=babababaababaabababababb and pat=ababababb.



3) i=i+MP_Shift(j)=

=i+j-Bord(j)=1+(7-5)=3

j=max(0,j-MP_Shift(j))=

=max(0,7-2)=5

pat[6]≠text[9]

4) i=i+MP_Shift(j)=

=3+MP_Shift(5)=3+(5-3)=5

j= max(0,j-MP_Shift(j))=

=max(0,5-2)=3

pat[4]≠text[9]

Morris-Pratt (MP) algorithm

- Lemma: The time complexity of the algorithm MP is linear in the length of the text (i.e., O(n)). The maximal number of character comparisons is 2n-m.

Proof:

- Let T(n) be the maximal number of comparisons pat[j+1]=text[i+j+1].

- There are at most n-m+1 unsuccessful comparisons: at most one for each i (0 ≤ i ≤ n-m)

- 0≤i+j≤n

Algorithm MP;

$i := 0;\ \ j := 0;$

while $i \le n - m$ do begin

while $j < m$ and

$pat[j+1] = text[i+j+1]$ do

$j = j+1;$

if $j = m$ then $return(true);$

$i := i + MP_Shift(j);$

$j := \max(0, j - MP_Shift(j));$

end;

$return(false)$

Morris-Pratt (MP) algorithm

- Each time we have a successful comparison we increase i+j by 1 and this value never decreases.
- So there are at most n successful comparisons.
- If the first comparison is unsuccessful, then we do not have any successful comparison for i=0.
- So, $T(n) \leq n+n-m=2n-m$.
- If text=a^n and pat=ab, then we actually have $T(n)=2n-m$

Algorithm MP;

$i := 0;\ \ j := 0;$

while $i \leq n-m$ do begin

  while $j < m$ and

   $pat[j+1] = text[i+j+1]$ do

    $j = j+1;$

  if $j = m$ then $return(true);$

  $i := i + MP_Shift(j);$

  $j := \max(0, j - MP_Shift(j));$

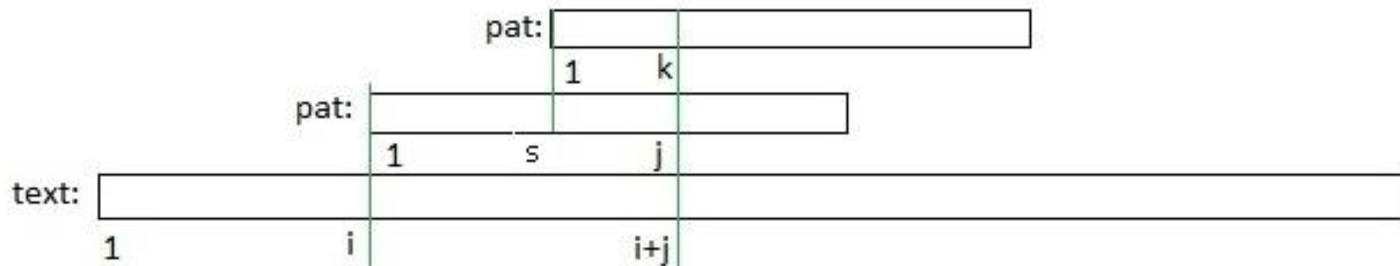
end;

$return(false)$

Unnecessary comparisons in the Morris-Pratt algorithm

- If we had a mismatch in the MP-algorithm , i.e., pat[j+1]≠text[i+j+1], then the next comparison is between text[i+j+1] and pat[k+1], where k=Bord(j).

- If pat[k+1]=pat[j+1], then we have the same mismatch.



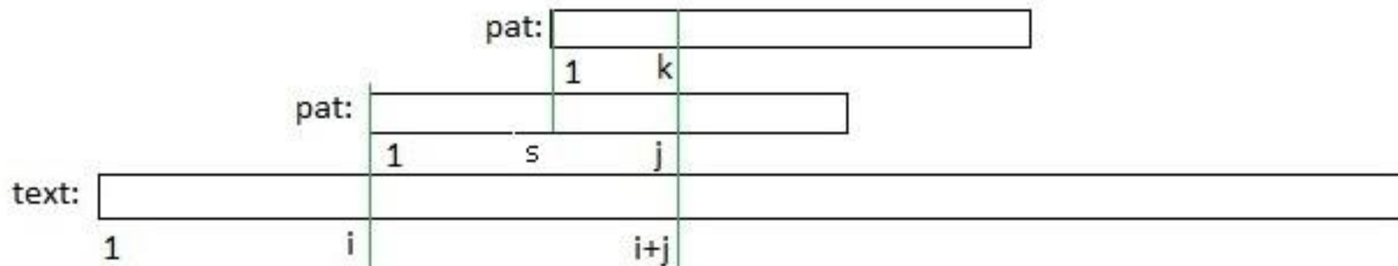
Unnecessary comparisons in the Morris-Pratt algorithm

- So, instead of the condition used in the MP algorithm:

  cond(j,k): pat[1..k] is a proper suffix of pat[1..j]

we can use a stronger one for $0 \leq j < m$:

  strong_cond(j,k): pat[1..k] is a proper suffix of pat[1..j] and pat[k+1]≠pat[j+1]



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Knuth-Morris-Pratt (KMP) algorithm

- We define Strong_Bord(j):
  - the largest integer k satisfying strong_cond(j,k),
  - -1 otherwise,
  - Strong_Bord(m)=Bord(m).

- Example: For the pattern pat=abaab we have the following functions Bord and Strong_Bord:

Bord	Strong_Bord
Bord(0)=-1	Strong_Bord(0)=-1
Bord(1)=0	Strong_Bord(1)=0
Bord(2)=0	Strong_Bord(2)=-1
Bord(3)=1	Strong_Bord(3)=1
Bord(4)=1	Strong_Bord(4)=0
Bord(5)=2	Strong_Bord(5)=2

Knuth-Morris-Pratt (KMP) algorithm

- The function MP_Shift from MP-algorithm is replaced by KMP_Shift(j)=j-Strong_Bord(j)

- For the new, improved algorithm, the table Strong_Bord becomes essential and is computed in the pre-processing step.

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Computation of the table Strong_Bord

- Let t=Bord(j)
- If pat[t+1]≠pat[j+1], then Strong_Bord(j)=Bord(j).
- Otherwise, i.e., pat[t+1]=pat[j+1], we have Strong_Bord(j)=Strong_Bord(t) (=Strong_Bord(Bord(j)))
- Strong_Bord(|pat|)=Bord(|pat|) (as if pat is followed by an end marker)

Computation of the table Strong_Bord

procedure compute_Strong_Bord

$Strong_Bord(0) := -1; \quad t := -1;$

for $j := 1$ to m do begin

while $t \geq 0$ and $pat[t+1] \neq pat[j]$ do $t := Strong_Bord(t)$;

$t := t+1;$

if $j = m$ or $pat[t+1] \neq pat[j+1]$ then $Strong_Bord(j) := t$;

else $Strong_Bord(j) := Strong_Bord(t)$;

end

- Example: For the pattern pat=aba^{m-2}, the table Strong_Bord computed by this algorithm is:

Strong_Bord(0)=-1, Strong_Bord(1)=0,
Strong_Bord(2)=-1, Strong_Bord(j)=1, for all 3≤j≤m.

Computation of the table Strong_Bord

procedure compute_Strong_Bord

$\text{Strong_}Bord(0) := -1; \quad t := -1;$

for $j := 1$ to m do begin

while $t \geq 0$ and $pat[t+1] \neq pat[j]$ do $t := Strong_Bord(t);$

$t := t + 1;$

if $j = m$ or $pat[t+1] \neq pat[j+1]$ then $Strong_Bord(j) := t;$

else $Strong_Bord(j) := Strong_Bord(t);$

end

- Lemma: The maximum number of character comparisons executed by algorithm compute_Border is 3m-5.

Knuth-Morris-Pratt (KMP) algorithm

- The algorithm KMP is the algorithm MP where Bord is replaced by Strong_Bord and MP_Shift is replaced by KMP_Shift

Algorithm KMP;

$i := 0; \;\; j := 0;$

while $i \le n - m$ do begin

while $j < m$ and $pat[j+1] = text[i+j+1]$ do

$j = j + 1;$

if $j = m$ then $return(true)$;

$i := i + KMP_Shift(j); \;\; j := \max(0, j - KMP_Shift(j));$

end;

$return(false)$

Knuth-Morris-Pratt (KMP) algorithm

- Lemma: The complexity of Knuth-Morris-Pratt algorithm is in $O(n)$.
  - The complexity of the searching phase of the Knuth-Morris-Pratt algorithm is in $O(n)$.
  - The pre-processing phase has a complexity of $O(m)$.
  - Since $m \leq n$, the overall complexity of the Knuth-Morris-Pratt algorithm is in $O(n)$.